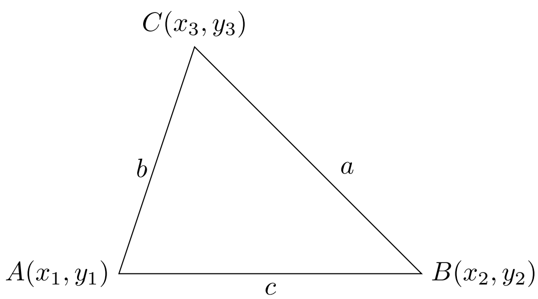
三角形的 31 个面积公式

天涯客
百度 ID: 天涯客 063

(本文档由 LaTeX 排版系统编辑而成，编辑环境：TeXLive 2014 + TeXstudio)



令 $p = \dfrac{1}{2}(a+b+c)$

$\triangle ABC$ 三边上的高分别为 h_a, h_b, h_c

$\triangle ABC$ 的外接圆半径为 R，内切圆半径为 r

AB, AC, BC 边上的中线分别是 m_c, m_b, m_a

$\angle A, \angle B, \angle C$ 的平分线长分别为 t_a, t_b, t_c

$\angle A, \angle B, \angle C$ 的外角平分线长分别为 t'_a, t'_b, t'_c

$\angle A, \angle B, \angle C$ 所对的旁切圆半径分别为 r_a, r_b, r_c

$$\begin{aligned}
&S_{\triangle ABC}\\
=&\frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c && (1)\\
=&\sqrt{p(p-a)(p-b)(p-c)} && (2)\\
=&pr && (3)\\
=&\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C && (4)\\
=&\left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}\right| && (5)
\end{aligned}$$

$$
\begin{aligned}
& S_{\triangle ABC} \\
=& \frac{1}{2}\sqrt{\left(\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AB}\right|\right)^2-\left(\overrightarrow{AC}\cdot\overrightarrow{AB}\right)^2} && (6)\\
=& \frac{a^2\sin B\sin C}{2\sin A}=\frac{b^2\sin A\sin C}{2\sin B}=\frac{c^2\sin A\sin B}{2\sin C} && (7)\\
=& 2R^2\sin A\sin B\sin C && (8)\\
=& \frac{h_c^2\sin C}{2\sin A\sin B}=\frac{h_a^2\sin A}{2\sin B\sin C}=\frac{h_b^2\sin B}{2\sin A\sin C} && (9)\\
=& \frac{abc}{4R} && (10)\\
=& \frac{a^2\sin 2B+b^2\sin 2A}{4}=\frac{b^2\sin 2C+c^2\sin 2B}{4}=\frac{a^2\sin 2C+c^2\sin 2A}{4} && (11)\\
=& \frac{(a^2-b^2)\sin A\sin B}{2\sin(A-B)}=\frac{(a^2-c^2)\sin A\sin C}{2\sin(A-C)}=\frac{(b^2-c^2)\sin B\sin C}{2\sin(B-C)} && (12)\\
=& \frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} && (13)\\
=& \left(\frac{a^2}{\sin A}+\frac{b^2}{\sin B}+\frac{c^2}{\sin C}\right)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} && (14)\\
=& \frac{1}{2}\sqrt{a^2b^2-\left(\frac{a^2+b^2-c^2}{2}\right)^2} && (15)\\
=& 4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} && (16)\\
=& 4Rp\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} && (17)\\
=& p(p-a)\tan\frac{A}{2}=p(p-b)\tan\frac{B}{2}=p(p-c)\tan\frac{C}{2} && (18)\\
=& p^2\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} && (19)\\
=& \frac{1}{8}(4m_a^2-a^2)\tan A=\frac{1}{8}(4m_b^2-b^2)\tan B=\frac{1}{8}(4m_c^2-c^2)\tan C && (20)\\
=& \frac{b+c}{2\sqrt{bc}}\sqrt{(p-b)(p-c)}t_a=\frac{a+c}{2\sqrt{ac}}\sqrt{(p-a)(p-c)}t_b=\frac{a+b}{2\sqrt{ab}}\sqrt{(p-a)(p-b)}t_c && (21)\\
=& \frac{t_at_bt_c}{abcp}(a+b)(b+c)(c+a) && (22)\\
=& \frac{|b-c|}{2\sqrt{bc}}\sqrt{p(p-a)}t'_a=\frac{|a-c|}{2\sqrt{ac}}\sqrt{p(p-b)}t'_b=\frac{|a-b|}{2\sqrt{ab}}\sqrt{p(p-c)}t'_c && (23)\\
=& \frac{\left|b^2-c^2\right|}{4bc}t_at'_a=\frac{\left|a^2-c^2\right|}{4ac}t_bt'_b=\frac{\left|a^2-b^2\right|}{4ab}t_ct'_c && (24)\\
=& rr_a\cot\frac{A}{2}=rr_b\cot\frac{B}{2}=rr_c\cot\frac{C}{2} && (25)
\end{aligned}
$$

$$
\begin{aligned}
&S_{\triangle ABC} \\
=&\sqrt{r r_a r_b r_c} && (26)\\
=&\frac{r_a r_b r_c}{\sqrt{r_a r_b + r_b r_c + r_c r_a}} && (27)\\
=&aR\sin B\sin C = bR\sin A\sin C = cR\sin A\sin B && (28)\\
=&\frac{1}{4}\left(b^2+c^2-a^2\right)\tan A = \frac{1}{4}\left(a^2+c^2-b^2\right)\tan B = \frac{1}{4}\left(a^2+b^2-c^2\right)\tan C && (29)\\
=&r_a(p-a) = r_b(p-b) = r_c(p-c) && (30)\\
=&\frac{a^2+b^2+c^2}{4(\cot A+\cot B+\cot C)} && (31)
\end{aligned}
$$